Lyapunov exponents of random time series

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(Received 20 November 1995; revised manuscript received 13 March 1996)

The conventional method for estimating Lyapunov spectra can give spurious positive Lyapunov exponents when applied to random time series. We analyze this phenomenon by using a simple stochastic model which produces completely random time series with no temporal correlation. We show that the possible estimation of spurious positive Lyapunov exponents is due to the statistical nature and finiteness of data. We also derive an upper bound of the largest Lyapunov exponent for the model. [S1063-651X(96)05608-5]

PACS number(s): 05.45.+b, 05.40.+j, 02.50.−r

In studying the mechanism that generates a fluctuating time series, it is important to determine whether or not the series is produced from a deterministic system with chaotic dynamics and, if it is, to characterize the dynamics. The Lyapunov spectrum gives information useful for characterizing orbital instability of chaotic dynamics, and in recent years it has been increasingly applied to time series analysis, along with various methods concerning it [1]. When we want to estimate the Lyapunov spectrum from a given time series we can use the customary method consisting of the phase space reconstruction via embedding and the local function approximation [2–4]. Knowing how this method behaves when it is applied to a chaotic time series contaminated by noise or applied (improperly) to random time series that are not chaotic would extend our understanding of the method and help to ensure that it is used properly.

Ikeguchi and Aihara [5] have reported that naive application of the method to random time series (time-interval data of γ-ray emission by cobalt) can, as expected, give spurious positive Lyapunov exponents. Since their result was based on numerical experiments, here we show analytically what kind of results we would obtain when applying the method blindly to a completely random time series.

It should be noted that the case in which the time series is completely random is itself of little practical importance because we can easily discriminate such a series from those resulting from deterministic chaos if we follow appropriate procedures, such as the false-nearest-neighbor test [6] and tests of determinism [7,8]. Thus we intend to discuss not the practical risk of misinterpretation of completely random time series as chaotic ones but a basic property of the method in an analytically tractable way.

We treat the following model: Consider a series of independent and identically distributed (iid) random variables $\{X_n\}$ with a probability density function which is uniform on a unit circle $R=\mathbf{R}/\mathbf{Z}$. We regard a realization of the series as an output time series from a fictitious deterministic dynamical system and analyze it by blindly applying the method of estimating the Lyapunov spectra.

The method treated in this report follows the processes of embedding time series with delay coordinates and of reconstructing dynamics as local linear maps by the least-mean-square approximation [2–4]. Our model has the advantage of simplicity, because the neighborhoods of all points in delay coordinates have exactly the same statistical properties. We

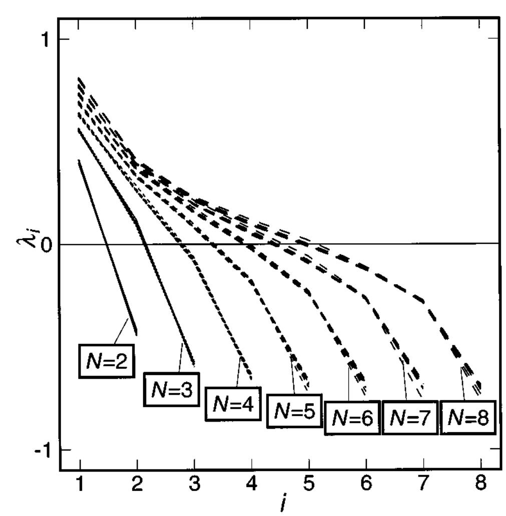

FIG. 1. Lyapunov spectra for various values of L and N ($m=100$, $\rho=0.01$). For each value of N, the results with the following values of L are plotted: 10, 20, 50, 100, 200, 500, 1000, 2000, 5000, and 10 000.

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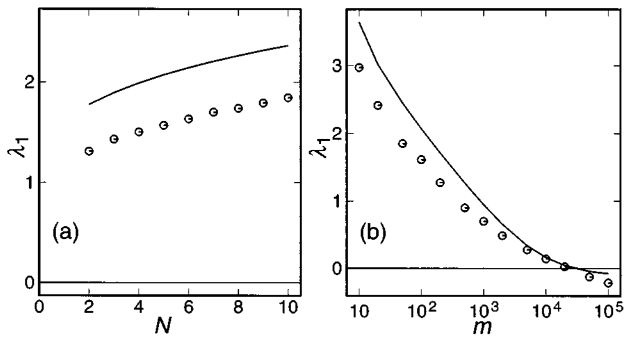

FIG. 2. Maximal Lyapunov exponent λ_1: (a) versus the embedding dimension N while m is fixed at 100, and (b) versus the number of neighboring data m while N is fixed at 5. Circles represent the simulation result and lines represent the result calculated from Eq. (2). $\rho = 0.01$ and $L = 10\,000$ for both cases.

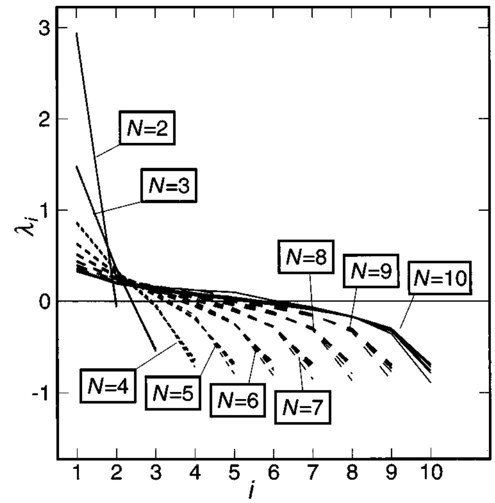

FIG. 3. Lyapunov spectra for various values of L and N [the total number of data is fixed to be 10^4, $m = \min(2N, N+4)$]. For each value of N, the results with the following values of L are plotted: 10, 20, 50, 100, 200, 500, 1000, 2000, 5000, and 10 000.

assume that the local linear maps are constructed from neighboring m data which are independently and uniformly distributed within a sphere of radius ρ in N-dimensional delay coordinates. Since the time series is random, the resulting maps are also random. From a sequence of the random matrices $\{P_i\}$ describing the random mappings we formally construct the Oseledec matrix

$$\mathbf{\Lambda}(L) = \left(\left[\prod_{i=1}^{L} P_i \right]^T \cdot \prod_{i=1}^{L} P_i \right)^{1/2L}, \tag{1}$$

where $\Pi_i P_i = P_L P_{L-1} \ldots P_2 P_1$. As in the case of the multiplicative ergodic theorem, we expect that the limit of $\mathbf{\Lambda}(L)$ as $L \to \infty$ exists for each sequence of the random matrices. The logarithm of the eigenvalues of the Oseledec matrix when $L \to \infty$ are denoted $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N$. We regard λ_i as the global Lyapunov exponents of the model. Practically, the limit $L \to \infty$ is evaluated by using sufficiently large L. We use the succesive decomposition [4,9] for the evaluation of the eigenvalues.

Figure 1 shows the ith largest Lyapunov exponent λ_i versus i for various values of L and the embedding dimension N. The results are affected very little by the values of L between 10 and 10 000. Figures 2(a) and 2(b) show the overall dependence of the largest Lyapunov exponent λ_1 on N and on m. For a fixed ρ, λ_1 is an increasing function of N and a decreasing function of m. The details of the simulation are given elsewhere [10].

Eckmann and Ruelle [3] suggest the use of $\min(2N, N+4)$ for the value of m, and Fig. 3 shows the result of the computer simulation for evaluating the use of this value. This result is very similar to that obtained with actual cobalt data [5].

The largest Lyapunov exponent is related to the averaged expansion rate of vectors by the random maps $\{P_i\}$. The averaged expansion rate e of P_i can be estimated to be

$$e = \left(\sigma^2 - \frac{1}{N} + 1 \right)^{1/2}, \tag{2}$$

where σ^2 is the variance of each component of the local linear predictor [10]. The derivation of Eq. (2) will be given elsewhere. Shown in Fig. 2 by solid lines are the values of $\ln\ e$, that is, the values of the largest Lyapunov exponent estimated on the basis of Eq. (2). It is estimated to be negative for large m since σ^2 tends to be zero as $m \to \infty$, as expected from its statistical property. In practical situations, however, we cannot take the limit $m \to \infty$ because the amount of data available is always finite. It can be shown that the principal correction term to this estimation is negative, and hence this estimation gives an upper bound of the largest Lyapunov exponent of the model.

In general, if the available data are of finite length, what we can say about whether or not it is chaotic must take a form of statistical rather than deterministic statements. Pecora, Carroll, and Heagy [11] have discussed a framework for statistical statements on chaotic time series in terms of mappings relating two data sets. As for the estimation of Lyapunov spectra our result gives an upper bound of the largest Lyapunov exponent which can be expected solely by the randomness of time series.

Our result has a direct practical application of estimating an upper bound of the largest Lyapunov exponent on random-shuffled surrogate data [12]. Moreover, the result suggests that when random contamination may be added to experimentally obtained chaotic time series, the Lyapunov spectrum analysis should be applied very carefully to avoid spurious estimation of positive Lyapunov exponents as evidences of deterministic chaos.

We thank T. Ikeguchi at the Faculty of Industrial Science and Technology, Science University of Tokyo, and N. Murata and H. Shimokawa at the Faculty of Engineering, the University of Tokyo, for their helpful comments.

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